FEBRUARY 12TH, 1849.

REV. HUMPHREY LLOYD, D. D., PRESIDENT, in the Chair.

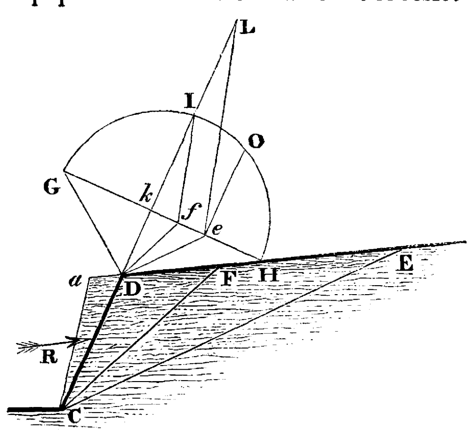
MAURICE Colles, Esq.; Rev. John Magrath, LL. D.; Jeremiah J. Murphy, Esq.; and William Ogilby, Esq.; were elected Members of the Academy.

Mr. M. Donovan continued the reading of his paper on electricity.

Sir William Betham read a paper on the feudal land tenures and dignities, and their introduction into Ireland at the English conquest.

The Rev. Charles Graves, on the part of the Rev. Brice Bronwin, presented a paper on the theory of planetary disturbance.

Mr. J. Neville read a paper on the maximum amount of resistance acting in any direction required to sustain banks of earth or other materials, with sloping tops and faces, and the effects of friction between the face of the bank and the back of a retaining structure.



If CDE be any bank with a sloping face CD, and a sloping top, DE; CE the position of the

plane of repose, CF that of the plane of fracture, and the arrow R that of the resistance: put

c' = the angle of repose.

c = the complement of the angle of repose.

β = the angle DCE contained between the plane of repose and the face of the bank.

δ = the supplement of the sum of the complement of the angle of repose, and the angle which the given direction of the resistance makes with the face.

θ = the angle KDF, contained between the face produced and the top of the bank.

ϕ = the angle DCF, contained between the plane of fracture and the face.

h = the length of the face CD.

w = the weight of a cubical unit of the bank.

R = the resistance.

Then, when the resistance is a maximum,

$$\tan\phi = \frac{\tan\beta\,\sqrt{(\tan\theta\tan\delta)}}{\sqrt{(\tan\theta\tan\delta)}+\sqrt{\{(\tan\beta+\tan\delta)\times(\tan\theta-\tan\beta)\}}}, \quad (1)$$

$$\left.\begin{aligned} &R = \frac{wh^2\tan\theta\sin\beta\tan\beta}{2\cos\delta} \\ &\left(\frac{1}{\sqrt{\{\tan\delta(\tan\theta-\tan\beta)\}}+\sqrt{\{\tan\theta(\tan\delta+\tan\beta)\}}}\right)^2. \end{aligned}\right\} \quad (2)$$

Equation (1) furnishes the following geometrical construction for finding the fracture CF. Draw any line GH at right angles to the face produced, cutting the slope DE at H and the line DG; making the angle GDK $=\delta$ at G: on GH describe a semicircle cutting the face produced in I: draw D*e* parallel to the plane of repose, CE, meeting GH in *e*: draw *e*O parallel to KI, meeting the circumference in O: make IL equal *e*O; draw I*f* parallel to L*e*, and CF parallel to D*f*; CF is the fracture requiring a maximum resistance to sustain the bank CDF.

If the top lying between F and D be loaded with a given weight, the values of ϕ and R are rigorously determined from the equations by producing the top ED to d, so that the triangle CDd, multiplied by w and the length of bank acted on, may be equal to the given weight, and then substituting the new values of h, δ, θ, and β, corresponding to the face Cd and top Ed, in the equations, in place of those to the face CD and top ED.

When the resistance is generated by the pressure of the bank against a structure at the face, δ may be taken equal $2c'$. In this case.

$$\tan\phi = \frac{\tan\beta\ \sqrt{(\tan\theta\ \tan 2c')}}{\sqrt{(\tan\theta\tan 2c')}+\sqrt{\{(\tan\theta-\tan\beta)\times(\tan\beta+\tan 2c')\}}}, \qquad (3)$$

$$\left.\begin{aligned} &\mathrm{R} = \frac{wh^2}{2}\ \frac{\sin\beta\ \tan\beta\ \tan\theta}{\cos 2c} \\ &\left(\frac{1}{\sqrt{\{(\tan 2c'(\tan\theta-\tan\beta)\}}1+\sqrt{\{\tan\theta\,(\tan 2c'+\tan\beta)\}}}\right), \end{aligned}\right\} \qquad (4)$$

When the face is vertical and the top horizontal, $c=\beta$: in this case

$$\tan\phi = \frac{\cos c'}{\sin c'+\sqrt{\tfrac{1}{2}}}, \qquad (5)$$

$$\mathrm{R} = \frac{wh^2\sec c'}{2}\left(\frac{1}{\sqrt{2}\ \tan c'+\sec c'}\right)^2. \qquad (6)$$

The value of $\tan\phi$ here derived is equivalent to that of $\dfrac{1}{\tan a}$ in equation (F) of Tredgold;* but the value of the resistance differs materially from his, and is far more simple. Tredgold's equation (G) for the value of the resistance acting horizontally, after making the necessary changes to our notation, is

$$\mathrm{R} = \frac{h^2w}{2}\times\frac{1}{\sin c'\sqrt{2}+1+\dfrac{\sin^3 c'\sqrt{2}+\sin^2 c'}{\cos^2 c'}+\dfrac{\sqrt{2}}{2\cos c'}}.$$

* Philosophical Magazine, vol. li. p. 402.

This value, however, is erroneous, and should be

$$R=\frac{h^2w}{2}\times\frac{1}{\sin c'\sqrt{2}+1+\dfrac{\sin^3 c'\sqrt{2}+3\sin^2 c'+\sin c'\sqrt{2}}{\cos^2 c'}},$$

which, multiplied by sec c', *to find the resulting* resistance, is equal to the more simple form found above.

When $\theta=\beta$ the top slopes upwards at the angle of repose: in this case

$$\tan\phi=\tan\beta, \tag{7}$$

$$R=\frac{wh^2}{2}\times\frac{\sin^2\beta}{\sin(2c'+\beta)}. \tag{8}$$

The second of these equations gives the greatest of the maximum values of the resistance: if the face be vertical, $\tan\beta=\dfrac{1}{\tan c'}$, and

$$R=\frac{wh^2}{2}\cos c'. \tag{9}$$

The horizontal portion of this resistance is

$$R=\frac{wh^2}{2}\cos^2 c'=\frac{wh^2}{2}\sin^2 c. \tag{10}$$

As this value is the same as (7*) the limiting value of the horizontal resistance, neglecting friction at the face, it appears that the limiting value of the horizontal resistance is the same whether friction at the face be taken in the calculation or neglected.

When the top slopes downwards at the natural slope,

$$\tan\phi=\tan\tfrac{1}{2}\beta, \tag{11}$$

$$R=\frac{wh^2}{2}\,\frac{\sin\beta\tan\beta}{\cos 2c'}\left(\frac{\sqrt{(\tan 2c'+\tan\beta}}{\tan 2c'\sec\beta+\tan 2c'+\tan\beta}\right)^2. \tag{12}$$

The value of the resistance here given is the least of the maximum values. If the face be vertical,

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* Proceedings, vol. iii. p. 86.

$$\tan \phi = \tan \tfrac{1}{2} c, \tag{13}$$

$$R = \frac{wh^2}{2} \sec c' \left(\frac{1}{2 \tan c' + \sec c'} \right)^2. \tag{14}$$

The value of the angle of fracture is of the same form as that of Prony for a vertical face and horizontal top.

The equations show that the stability imparted to a structure at the face of a bank, by friction, arises principally from the direction of the resulting force, which makes an angle equal to the complement of the angle of repose with the face, and that this force is in general less than the horizontal force derived from the equation of Prony, or any other in which face friction is neglected ; that the values of both forces, for ordinary banks, are equal at angles of repose in and about 45°; that the former are least for angles of repose less than this, and the latter for angles of repose that are greater ; and that the direction of the resulting force makes it in no small degree a crushing force.

It also appears from the equations, that when the angle of repose is 45°, the face vertical, and top horizontal, that the tangent of the angle of fracture is ($\frac{1}{2}$) equal half the tangent of the angle of repose. The equation of Prony, for the same case, gives the tangent of the angle of fracture equal to the tangent of half the angle of repose.

In the following Table of Coefficients, for finding the maximum values of the resistances,

Column 1 contains the engineering names for the slopes corresponding to some of the angles of repose in Column 2.

Column 2 contains the angles of repose from which the coefficients of wh_1^2 are calculated.

Column 3 contains the complements of the angles of repose in column 2 ; or the angle which the direction of the *resulting resistance* makes with the face, taking friction thereof into account.

Column 4 contains the coefficients which, multiplied by

wh_1^2, give the value of the horizontal resistances when the top is horizontal and the face vertical; calculated from the "Equation of Prony."

Column 5 contains the coefficients which, multiplied by wh_1^2, give the values of the horizontal resistances, rejecting friction at the face, required to sustain banks with a horizontal top; the face sloping 10° from the vertical: $\theta = 80°$.

Column 6 contains the coefficients which, multiplied by wh_1^2, give the values of the *resulting resistances* when the top is horizontal and the face vertical, as in Column 4.

Column 7 contains the values of the coefficients, as before, for finding the *resulting resistances* when the top is horizontal and the face slopes 10° from the vertical, as in Column 5: $\theta = 80°$.

Column 8 contains the values of the coefficients for finding the values of the *resulting resistances* when the face overhangs 10° from the vertical, and the top is horizontal: in this case $\theta = 100°$.

Column 9 contains the resolved coefficients of wh_1^2 for finding the portions of the resistances in Column 6 at right angles to the face, which in this case are horizontal.

Column 10 contains the resolved coefficients of wh_1^2 for finding the portions of the resistances in Column 7 at right angles to the face. These, in this case, not differing much from the resolved horizontal portions, may be compared with those in Column 5.

Column 11 contains the resolved coefficients of wh_1^2, for finding the portions of the resistances in Column 8 at right angles to the face.

Column 12 contains the values of the coefficients which, multiplied by wh_1^2, give the ultimate or *maximum maximorum* values of the *resulting resistances;* the face being vertical and the top sloping upwards, at the slope of repose.

Column 13 contains the coefficients for finding the horizontal portions of the resistances determined from Column 12.

The length of the perpendicular from the toe of the face to the top, or top produced, is represented by h_1; and the length of the face itself by h.

wh_1^2 is multiplied by the coefficients in Columns 4 to 11 inclusive, to find the resistances; and wh^2 by the coefficients in Columns 11 and 12.

TABLE *of Coefficients for finding the maximum Values of the Resistances for different Angles of Repose; also the Coefficients for finding the ultimate Values of the Resistances when the Face is vertical, and Scarp at the natural Slope.*

1	2	3	4	5	6	7	8	9	10	11	12	13
3½ to 1*	16°	74°	.284	.228	.249	.218	.287	.239	.209	.276	.481	.462
	17	73	.274	218	.239	.207	.271	.228	.198	.259	.478	.457
3 to 1*	18½	71½	.259	.207	.226	.193	.266	.214	.183	.262	.474	.450
2½ to 1*	22	68	.228	.177	.197	.164	.210	.183	.152	.195	.464	.430
2 to 1*	27	63	.188	.141	.165	.130	.208	.147	.116	.185	.446	.397
	29	61	.173	.129	.155	.119	.197	.136	.104	.172	.437	.382
	31	59	.160	.117	.144	.108	.188	.123	.093	.161	.429	.367
	32	58	.153	.111	.139	.103	.183	.118	.087	.155	.424	.360
	33	57	.147	.106	.134	.098	.177	.113	.082	.149	.419	.352
1½ to 1*	34	56	.141	.101	.129	.094	.173	.107	.078	.143	.415	.344
	35	55	.135	.095	.126	.090	.169	.103	.074	.138	.410	.336
	36	54	.130	.090	.121	.086	.165	.098	.070	.133	.405	.327
	37	53	.124	.084	.117	.081	.161	.093	.065	.129	.400	.319
	39	51	.114	.077	.108	.074	.154	.084	.057	.120	.389	.302
	41	49	.104	.069	.102	.067	.146	.077	.051	.110	.327	.284
	43	47	.094	.062	.095	.061	.140	.069	.045	.102	.366	.267
1 to 1	45	45	.085	.054	.089	.055	.134	.063	.039	.095	.354	.250
	47	43	.077	.048	.083	.049	.129	.057	.033	.088	.341	.233
	49	41	.070	.042	.077	.044	.123	.051	.029	.081	.328	.215
	51	39	.062	.036	.072	.039	.118	.045	.025	.074	.315	.198
¾ to 1*	53	37	.056	.031	.066	.035	.113	.040	.021	.068	.301	.181
	55	35	.049	.027	.062	.031	.109	.036	.018	.062	.287	.165
	57	33	.048	.022	.057	.027	.105	.031	.015	.057	.272	.149

The slopes marked thus * are approximate.

In the preceding equations we have only considered the maximum *retaining-forces*. The minimum *overcoming-forces*, and the position of the corresponding fractures, are determined in a similar manner, and by similar equations. Retaining the

same notation as before, we get, in this case, for the value of the *overcoming-force*,

$$R = \frac{why}{2} \times \frac{\sin(2c' + \beta - \phi)}{\sin(\delta - 2c' + \phi)}.$$

Where y is equal the perpendicular from (F) on the face, or face produced.

If we put

$$\beta_1 = 2c' + \beta,$$

and

$$\delta_1 = \delta - 2c';$$

the above equation, after a few reductions, becomes

$$R = \frac{why}{2} \times \frac{\cos\beta_1}{\cos\delta_1} \times \frac{\tan\beta_1 - \tan\phi}{\tan\delta_1 + \tan\phi}. \tag{15}$$

When this is a minimum,

$$\tan\phi = \frac{\tan\beta_1 \sqrt{(\tan\theta \tan\delta_1)}}{\sqrt{(\tan\theta\tan\delta_1)} - \sqrt{\{(\tan\theta - \tan\beta_1) \times (\tan\beta_1 + \tan\delta_1)\}}}, \tag{16}$$

$$\left.\begin{aligned} R = {} & \frac{wh^2 \tan\theta \sin\beta_1 \tan\beta_1}{2\cos\delta_1} \\ & \left(\frac{1}{\sqrt{\{\tan\theta(\tan\beta_1 + \tan\delta_1)\}} - \sqrt{\{\tan\delta_1(\tan\theta - \tan\beta_1)\}}}\right)^2; \end{aligned}\right\} \tag{17}$$

in which the usual changes of signs are to be made for the negative values of δ_1, and for arcs greater than 90°.

When the direction of the force makes an angle equal to c with the face, then $\delta_1 = 0$, and,

$$\phi = 0, \tag{18}$$

$$R = \frac{wh^2}{2}\sin\beta_1. \tag{19}$$

If the force exceed the value of R here found, it will slide along the face, and when the face is vertical this value is equal to the *maximum maximorum* value of the resistance, in the same case, already found; or,

$$R = \frac{wh^2}{2} \sin c.$$

When $\theta = 90°$, the general equations become

$$\tan \phi = \frac{\tan \beta_1 \sqrt{(\tan \delta_1)}}{\sqrt{(\tan \delta_1)} - \sqrt{\{(\tan \beta_1 + \tan \delta_{11})\}}}, \qquad (20)$$

$$R = \frac{wh^2 \sin \beta_1 \tan \beta_1}{2 \cos \delta_1} \left(\frac{1}{\sqrt{(\tan \beta_1 + \tan \delta_1)} - \sqrt{(\tan \delta_1)}}\right)^2. \qquad (21)$$

If the force in this case be supposed to act horizontally ($\delta_1 + \beta_1 = 90°$), these equations may be reduced to

$$\tan \phi = \cot \left(c - \frac{\beta}{2}\right); \qquad (22)$$

$$R = \frac{wh^2}{2} \cot^2 \left(c - \frac{\beta}{2}\right). \qquad (23)$$

If the face be vertical, then $\beta = c$, and the equations may be further reduced to

$$\tan \phi = \cot \tfrac{1}{2}c; \qquad (24)$$

$$R = \frac{wh^3}{2} \cot^2 \tfrac{1}{2}c. \qquad (25)$$

The Rev. Charles Graves communicated the following note respecting geodetic lines on surfaces of the second order.

"At a meeting of the Academy which took place in last June, I stated a general theorem, from which I am able to deduce Joachimsthal's theorem respecting the geodetic lines traced on a central surface of the second order; and at the same time to show geometrically the reason why the property enunciated in it is common to geodetic lines and to lines of curvature. From the general theorem to which I refer, the following proposition is a corollary:

"*If a central surface of the second order* (A) *be circumscribed by a cone* (a), *the quantity* PD *is the same for* L, L′, L″, L‴, *four sides of the cone which make equal angles with its internal axis:* P *denoting the perpendicular from the centre*